**On radiation reaction and the $[x,p]$ commutator for an accelerating charge.**

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**Abstract**

We formally state the connection between the relativistic part of the radiation reaction and the Poynting Robertson force term, $-Rv/c^2$, where $R$ is power radiated. Then we address the question, does $[x,p] = i\hbar$ for an accelerating charge ? The full radiation reaction term is used, which includes the relativistic term (von Laue vector.). We show that the full relativistic radiation reaction term must be taken into account if a commutation relation between $x$ and $p$ is to hold for an electron under uniform acceleration, consistent with the expectation values of $x^2$ and $p^2$.

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# Introduction

It has been shown [1], that the nonrelativistic theory of the electron is fundamentally inconsistent unless both radiation reaction and the vacuum field are allowed to act on the electron. In particular, it was shown that the canonical commutation rule $[x,p] = i\hbar$ for the electron is violated if radiation reaction is ignored. In this previous work [1], the equation of motion used was that of Lorentz.

$$\ddot{r}(t) - \gamma \dddot{r}(t) = \frac{e}{m} E_0(t) \tag{1}$$

where $\gamma = 2e^2/3mc^3$ and $r$ is a vector. We instead use the relativistic LAD equation of motion (of Lorentz, Abraham and Dirac) [2, 3, 4] given by

$$\ddot{x}_\nu - \gamma(\dddot{x}_\nu + \ddot{x}^\mu \ddot{x}_\mu \dot{x}_\nu / c^2) = \frac{e}{m} \dot{x}_\mu F^\mu_\nu \tag{2}$$

where the damping term $\gamma = 2r_0/3c$ is defined above and has dimensions of time, and $r_0 = e^2/mc^2$ is the classical electron radius. The term on the right is equivalent to $eE(t)/m$ where $E(t)$ is the electric

field. Also $m$ is the rest mass, which we shall use throughout the paper.

Let us examine this equation (2) a little more closely. The first term clearly comes from the kinetic energy of the charged particle (electron in this case). The second term on the left is the Abraham Lorentz radiation reaction term $F = 2e^2\dot{a}/3c^3$ or it is derived from what used to be called the acceleration energy by Schott 1915 [5]. The third term on the left is of the form $F = -Rv/c^2$ where $R = 2e^2a^2/3c^3$ which is the regular Larmor power radiated by an electron of acceleration $a$. This term corresponds exactly to the Poynting–Robertson force term [6, 7]. This is exactly the Abraham result of 1903 [3] and the relativistic part of the von Laue 1909 radiation reaction $\Gamma$ [8]. A history of the Dirac equation (2) is given by Rohrlich [9] and the book by Grandy [10]. The remarkable thing about Grandy's book is that on page 204 he writes the drag term (in the LAD equation) in the format of the Poynting–Robertson drag $-Rv/c^2$, where $R$ is the power radiated by the accelerating charge. However, no connection is made between the (relativistic) radiation reaction term and the Poynting–Robertson drag force. The author feels that this is an oversight, no electromagnetic text book mentions the connection even though the relativistic term is very important and used all the time, possibly without people realising it. The two radiation reaction force terms are both derived from the same power expression. Usually text books (for example Jackson [11] or Griffiths [12]) do not derive force expressions. They typically start with the electric field of a moving charge, then calculate the Poynting vector and then integrate over a cross–sectional area to find the power radiated into a given angle. Very few text books even mention the famous Dirac paper of 1938. Jackson's book refers to it in the problems section only, Griffiths' book does not mention it at all. It should also be mentioned that Dirac's 1938 paper contains the original discussion of the advanced and retarded fields, which was elaborated on by Wheeler and Feynman later in 1945.

When acceleration is taken into account, it is most important to keep the relativistic radiation reaction term. Boyer [13], noted that an electric dipole accelerated through the vacuum would see a surrounding field not quite equal to the usual Planck distribution. A correction term was needed [14], which turned out to be exactly the relativistic radiation reaction term, (or the Poynting and Robertson drag). Boyer [14], showed that a

> " classical electric dipole oscillator accelerating though classical electromagnetic zero–point radiation responds just as would a dipole oscillator in an inertial frame in a classical thermal radiation with Planck's spectrum at temperature $T = \hbar a/2\pi ck$"

where $T$ is the Unruh–Davies temperature.

Later, Boyer [15], did a similar calculation for the spinning magnetic dipole and found a mismatch with the Planck distribution again. He later corrected the magnetic dipole work with a similar drag force to regain the Planck distribution, [16]. He was able to show that,

> " the departure from Planckian form is canceled by additional terms arising in the relativistic radiative damping for the accelerating dipole. Thus the accelerating dipole behaves at equilibrium as though in an inertial frame bathed by exactly Planck's spectrum including zero–point radiation."

We have established that for an accelerated charge, consistency with thermodynamics requires that we keep the relativistic radiation reaction term in the equation of motion of the electron. We now turn our attention to the $[x, p]$ calculation. Instead of the quantized vacuum field, we will be dealing with the vacuum field and the Planck distribution at a temperature given by the Unruh Davies temperature $T = \hbar a/(2\pi c k_{\mathrm{B}})$ where $a$ is the uniform acceleration and $k_{\mathrm{B}}$ is Boltzmann's constant [14, 17].

We are strongly motivated by Wigner [18], who once asked the question;

> "Do the equations of motion determine the quantum mechanical commutation relations? "

Wigner showed that the equations of motion for a harmonic oscillator do not uniquely determine the commutation relations. We find that indeed the equation of motion in our case does determine the commutation relation for $x$ and $p$. The commutation relation found is consistent with the expectation values of $x^2$ and $p^2$ but at first sight it appears it is not $[x, p] = i\hbar$. Our commutation relation does however reduce to this expected result for the case when the acceleration of the charge $a \to 0$.

## Results and Discussion

Consider an electron undergoing an oscillation, like a mass on a spring, with the motion restricted to the $x$ direction. We shall take the resonant frequency of this oscillation to be $\omega_0$. This oscillatory motion is taken to be nonrelativistic and hence we continue to use the rest mass of the electron $m$. We consider the amplitude of this motion to be very small (as in the small dipole approximation) and we only keep terms linear in the amplitude $x$ of the oscillation. The electron is accelerating with uniform acceleration $a$ in the $z$ direction. We may rewrite the equation of motion in this case as,

$$\ddot{x} + \omega_0^2 x - \gamma\left(\dddot{x} - \frac{a^2}{c^2}\dot{x}\right) = \frac{e}{m}E(t) \tag{3}$$

where $E(t)$ is taken to be the electric field at the equilibrium point of the oscillation, instantaneously at rest with respect to the oscillation, as in Ref. [14]. Note that we have ignored terms in the radiative damping that go like $\ddot{x}^2\dot{x}$ since these would involve the amplitude of the oscillation squared which is a small quantity. We have only kept the $z$–component of the acceleration squared term because this term is only linear in the oscillation amplitude. We will write the electric field, (in the rest frame of the equilibrium point of the oscillation), in quantized form as

$$E(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty d\omega \; \epsilon_{ks} \left( \varepsilon(\omega_k) a_{ks} e^{-i\omega_k t} + \varepsilon^\star(\omega_k) a^\dagger{}_{ks} e^{i\omega_k t} \right) \tag{4}$$

where $a_{ks}$ is the photon annihilation operator for mode $(\vec{k}, s)$ and $\epsilon_{ks}$ is the polarization unit vector. The Fourier components $\varepsilon(\omega)$ are well defined by Boyer [14] and in earlier work by the same author cited therein. We leave the definition until later. Note also that we have taken $1/\sqrt{\pi}$ and not $1/\sqrt{2\pi}$ since that would imply an integral from $-\infty$ to $\infty$. Here we wish to keep the positive and negative frequency components separate. Let

$$x(t) = x_1 e^{-i\omega_k t} + x_2 e^{i\omega_k t} \tag{5}$$

and treat the positive and negative frequency terms separately in the equation of motion. Substituting $x_1$ into our Eq.(3) gives

$$x_1 = -\frac{e}{\sqrt{\pi} m} \int_0^\infty d\omega_k \; \frac{\epsilon_{ks} \varepsilon(\omega_k) a_{ks}}{(\omega_k^2 - \omega_0^2) + i\gamma(\omega_k^3 + a^2\omega_k/c^2)} \tag{6}$$

similarly for $x_2$ we find

$$x_2 = -\frac{e}{\sqrt{\pi} m} \int_0^\infty d\omega_k \; \frac{\epsilon_{ks} \varepsilon^\star(\omega_k) a^\dagger{}_{ks}}{(\omega_k^2 - \omega_0^2) - i\gamma(\omega_k^3 + a^2\omega_k/c^2)} \tag{7}$$

thus

$$x(t) = -\frac{e}{\sqrt{\pi} m} \int_0^\infty d\omega_k \; \epsilon_{ks} \left[ \frac{\varepsilon(\omega_k) a_{ks} e^{-i\omega_k t}}{(\omega_k^2 - \omega_0^2) + i\gamma(\omega_k^3 + a^2\omega_k/c^2)} + \frac{\varepsilon^\star(\omega_k) a^\dagger{}_{ks} e^{i\omega_k t}}{(\omega_k^2 - \omega_0^2) - i\gamma(\omega_k^3 + a^2\omega_k/c^2)} \right] \tag{8}$$

From this result we can easily find

$$\dot{x}(t) = i\frac{e}{\sqrt{\pi} m} \int_0^\infty d\omega'_k \; \epsilon_{k's'} \omega'_k \left[ \frac{\varepsilon(\omega'_k) a_{k's'} e^{-i\omega'_k t}}{(\omega'^2_k - \omega_0^2) + i\gamma(\omega'^3_k + a^2\omega'_k/c^2)} - \frac{\varepsilon^\star(\omega'_k) a^\dagger{}_{k's'} e^{i\omega'_k t}}{(\omega'^2_k - \omega_0^2) - i\gamma(\omega'^3_k + a^2\omega'_k/c^2)} \right] \tag{9}$$

The commutator then becomes

$$\begin{aligned}
[x, p] &= [x, m\dot{x}] \\
&= \frac{ie^2}{\pi m} \int_0^\infty d\omega_k \int_0^\infty d\omega'_k \; \epsilon_{ks} \cdot \epsilon_{k's'} \varepsilon(\omega_k) \varepsilon^\star(\omega'_k) \omega'_k \Bigg[ \\
&\quad \frac{[a_{ks}, a^\dagger{}_{k's'}] e^{-i(\omega_k - \omega'_k)t}}{[(\omega_k^2 - \omega_0^2) + i\gamma(\omega_k^3 + a^2\omega_k/c^2)][(\omega'^2_k - \omega_0^2) - i\gamma(\omega'^3_k + a^2\omega'_k/c^2)]} \\
&\quad - \frac{[a^\dagger{}_{ks}, a_{k's'}] e^{i(\omega_k - \omega'_k)t}}{[(\omega_k^2 - \omega_0^2) - i\gamma(\omega_k^3 + a^2\omega_k/c^2)][(\omega'^2_k - \omega_0^2) + i\gamma(\omega'^3_k + a^2\omega'_k/c^2)]} \Bigg]
\end{aligned} \tag{10}$$

We have assumed here it is sufficient to use, $[a_{ks}, a^{\dagger}{}_{k's'}] = \delta_{ss'}\delta^3_{kk'}$ and $[a^{\dagger}{}_{ks}, a_{k's'}] = -\delta_{ss'}\delta^3_{kk'}$ we find by setting $\omega_k = \omega$ (we drop the $k$ subscript which is no longer needed),

$$[x,p] = \frac{ie^2}{\pi m}\int_0^\infty d\omega\,\frac{2\omega\langle\varepsilon(\omega)\varepsilon^\star(\omega)\rangle}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^6[1+(a/c\omega)^2]^2} \tag{11}$$

At this point we introduce the expectation of the fields given by Boyer [14], in Eq.(25) of that paper.

$$\begin{aligned}\langle\varepsilon(\omega)\varepsilon^\star(\omega)\rangle &= \frac{4\hbar\omega^3}{3c^3}\left[1+\left(\frac{a}{c\omega}\right)^2\right]\coth\left[\frac{\pi c\omega}{a}\right]\\ &= \frac{8\pi^2}{3c}\left[1+\left(\frac{a}{c\omega}\right)^2\right]\rho(\omega)\coth\left[\frac{\pi c\omega}{a}\right]\end{aligned} \tag{12}$$

where $\rho(\omega) = \hbar\omega^3/(2\pi^2c^2)$ the energy density of free space. It is important to note at this stage that

$$\begin{aligned}\frac{1}{2}\coth\left[\frac{\pi c\omega}{a}\right] &= \frac{1}{2}\left(\frac{e^{2\pi c\omega/a}+1}{e^{2\pi c\omega/a}-1}\right)\\ &= \left(\frac{1}{2}+\frac{1}{e^{2\pi c\omega/a}-1}\right)\end{aligned} \tag{13}$$

where $2\pi c\omega/a = \hbar\omega/(k_{\rm B}T)$ which implies that $T = \hbar a/(2\pi ck_{\rm B})$ which is the Unruh Davies temperature. The $\rho(\omega)\coth$ term is therefore the vacuum field plus the Planck distribution at temperature $T$. Substituting the Eq.(12) into the integrand of Eq.(11) the commutator becomes,

$$[x,p] = \frac{4i\hbar}{\pi}\int_0^\infty d\omega\frac{\gamma\omega^4\left[1+\left(\frac{a}{c\omega}\right)^2\right]\coth\left[\frac{\pi c\omega}{a}\right]}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^6\left[1+\left(\frac{a}{c\omega}\right)^2\right]^2} \tag{14}$$

Then using the substitution $z = \omega^2-\omega_0^2$ and $s = a/(\omega_0 c)$ and setting $\omega=\omega_0$ everywhere else we get,

$$\begin{aligned}[x,p] &= \frac{2i\hbar}{\pi}\int_0^\infty dz\frac{\gamma\omega_0^3(1+s^2)\coth\left[\pi/s\right]}{z^2+\gamma^2\omega_0^6(1+s^2)^2}\\ &= \frac{2i\hbar}{\pi}\coth\left[\pi/s\right]\int_0^\infty dz\frac{A}{z^2+A^2}\end{aligned} \tag{15}$$

$$\tag{16}$$

where $A = \gamma\omega_0^3(1+s^2)$. Hence,

$$\begin{aligned}[x,p] &= \frac{2i\hbar}{\pi}\coth\left[\pi/s\right]\left[\tan^{-1}(z/A)\right]_0^\infty\\ &= i\hbar\coth\left[\pi c\omega_0/a\right]\end{aligned} \tag{17}$$

where we have substituted back for $s$ in the last step. This is not quite the expected result, but we do know that when $a\to 0$ then $\coth[\infty] = 1$ and we get back the expected result. To show that this is

indeed the correct result we need to consider the uncertainty relation, $\Delta x \Delta p$. Using,

$$\begin{aligned}\Delta x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\ \Delta p &= m\sqrt{\langle \dot{x}^2 \rangle - \langle \dot{x} \rangle^2}\end{aligned} \tag{18}$$

and $\langle x \rangle \equiv \langle \dot{x} \rangle = 0$, using the results from Boyer [14] we find;

$$\begin{aligned}\Delta x \Delta p &\geq m\sqrt{\langle x^2 \rangle \langle \dot{x}^2 \rangle} \\ &\geq \frac{\hbar}{2}\coth\left[\pi c \omega_0 / a\right]\end{aligned} \tag{19}$$

Generally speaking, we know from quantum mechanics text books, that if we have the uncertainty relation $\Delta a \Delta b \geq |d|/2$ then this implies a commutator $[a, b] = id$. In our case above $d = \hbar \coth\left[\pi c \omega_0 / a\right]$.

## Conclusions

In conclusion, we have made the connection between the relativistic radiation reaction term (in the LAD equation of motion) and the Poynting–Robertson drag force term. We have further shown that it is necessary to use the full equation of motion given by Lorentz, Abraham and Dirac (LAD) when treating an accelerated electron in order that the commutation relation (which agrees with the corresponding uncertainty relation) between $x$ and $p$ hold. The result reduces to the expected result when the acceleration $a \to 0$.

We also note that we have assumed a form of electric field quantization without a rigorous derivation. Our end result does appear consistent with previous work. However, we note that to regain the regular commutation relation it may be necessary to consider the relativistic Dirac equation, and use a combination of $\alpha$ or $\gamma$ matrices rather than $x$ and $p$ operators.

Finally, we note that is it possible to derive the result quickly by using the vacuum thermofield operators originally defined by Takahashi and Umezawa [19, 20] for the $a$ and $a^\dagger$ operators which then gives the $\coth\left[\pi c \omega_0 / a\right]$ term directly in the commutator. The thermofield commutator for $[a_{ks}, a^{\dagger}{}_{k's'}]_T = \coth\left[\pi c \omega_0 / a\right] \delta_{ss'} \delta^3_{kk'}$ would replace the commutator above Eq.(11), where we have used the Unruh Davies temperature $T$.

The Thermofield operators use a vacuum double Hilbert space, with a fictitious mode or virtual photon mode for the vacuum. The ordinary $a$ and $a^\dagger$ operators are transformed via a Bogoliubov transformation

from a unitary operator $T(\theta)$. The thermofield vacuum is given by,

$$|0\rangle_T = T(\theta)|0\tilde{0}\rangle \tag{20}$$

where $T(\theta) = \exp[-\theta(a\tilde{a} - a^\dagger\tilde{a}^\dagger)]$, where the tilde denotes a fictitious or virtual mode. Note that $T^\dagger(\theta) = T(-\theta)$. Using

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + ..., \tag{21}$$

we find,

$$\begin{aligned} a_T = T(\theta)aT^\dagger(\theta) &= a\cosh\theta - \tilde{a}^\dagger \sinh\theta \\ a^\dagger{}_T = T(\theta)a^\dagger T^\dagger(\theta) &= a^\dagger \cosh\theta - \tilde{a}\sinh\theta. \qquad (22) \\ & \qquad (23) \end{aligned}$$

It is easy to show that the photon number and commutator for the thermofield operators become,

$$\begin{aligned} \langle 0\tilde{0}|a^\dagger{}_T a_T|0\tilde{0}\rangle &= \sinh^2\theta = \frac{1}{e^{2\alpha} - 1} \\ \langle 0\tilde{0}|[a_T, a^\dagger{}_T]|0\tilde{0}\rangle &= \sinh^2\theta + \cosh^2\theta = \coth\alpha \qquad (24) \\ & \qquad (25) \end{aligned}$$

where $\alpha = \pi\omega_0 c/a$. Clearly we could have used these for our accelerating frame operators and saved a bit of time calculating electric fields like Boyer [14]. It is not clear if the thermofield operation can be related to space-time curvature, and quantum gravity, but the accelerated frame appears to give the same result as would the thermal frame using thermofield operators and the Unruh Davies temperature relation. It may be that we should simply normalize the $[a, a^\dagger]$ commutator to account for the thermal result, $\coth[\pi c\omega_0/a]$, in which case you regain the usual $[x, p]$ commutator. This would also normalize the uncertainty relation and we would get back $\Delta x\Delta p \geq \hbar/2$.


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